FULL DESCRIPTION OF THE PROJECT

1. Objectives

The *dimension* of a poset, introduced in 1941 by Dushnik and Miller [8], is arguably the most important measure of poset complexity. Computing dimension is a classical computationally hard problem. The decision problem whether $\dim(P) \leqslant k$, for a fixed k, was one of the twelve problems listed in 1979 by Garey and Johnson [14] that were not known to be solvable in polynomial time or NP-complete at that time.

Our first main objective is to understand when and why determining the dimension is computationally hard. Already in 1982, Yannakakis [48] showed that it is NP-hard even for posets of height 2. On the other hand, the following question remains open with basically no progress over many years:

(A) *Is there a polynomial-time algorithm to determine the dimension of a poset of bounded width?*

Resolving this question is the ultimate goal for the computational complexity track of our project. Already a stronger fixed-parameter tractability question is exciting and open:

(B) *Is there an algorithm computing the dimension of a poset of size n and width w in time $f(w)n^{\alpha}$ for some function f and constant α?*

Posets are visualized by their diagrams, which represent those order relations which are minimal in the sense that they are not implied by transitivity. A poset is *planar* if its diagram can be drawn in the plane with no edge crossings so that the order relation of any two comparable points agrees with their y-coordinate order in the drawing. The *cover graph* of a poset is its diagram considered as an abstract graph with no geometric restrictions on its possible drawings. The second line of research follows a common belief that when a poset has a 'nice drawing' or 'not too dense' cover graph, then it should have small dimension.

Already in the 1970s, Trotter and Moore [45] proved that posets with cover graphs being forests (excluding K_3 as a minor) have dimension at most 3. Only very recently we have proved [23] that posets with cover graphs excluding K_4 as a minor have constant dimension. It is well known that planar posets may have arbitrarily large dimension. These can be all considered as partial results towards the exact characterization of which excluded minors yield a constant bound on the dimension.

(C) *For which minor-closed classes of graphs $\mathscr{C}$ is it true that posets with cover graphs in $\mathscr{C}$ have dimension bounded by a constant?*

An important recent breakthrough in dimension theory was a result by Streib and Trotter [41] that posets with planar cover graphs have dimension bounded by a function of height. The function resulting from their proof is enormous, and they did not even hope that one can get a good bound. Now the best known bound, coming from our earlier work [32], is double exponential in height. We feel that we are close to the positive resolution of the following:

(D) *Do planar posets have dimension bounded by a linear function of the height?*

If this is true, then it will open a new track of research which hopefully will lead us to techniques giving precise bounds for dimension.

The initial result bounding dimension of posets with planar cover graphs in terms of their height launched a flurry of more general positive results, replacing the planarity condition for cover graphs by bounded treewidth, excluded (topological) minor, and very recently bounded expansion (in the sense of Nešetřil-Ossona de Mendez sparsity theory). For a complete picture of the behavior of dimension for sparse classes of graphs, one more statement, suggested to us by Daniel Král' and very likely to be true, remains unproven:

(E) *For any nowhere dense class $\mathscr{C}$ of graphs and any $\varepsilon > 0$, is it true that posets of size n with bounded height and with cover graphs in $\mathscr{C}$ have dimension $O(n^{\varepsilon})$?*

This would be the analogue of the fact that chromatic number of graphs from a nowhere dense class is $O(n^{\varepsilon})$.

Yet another interesting research direction inspired by the result that planar posets with bounded height have bounded dimension is to relax (or replace) the bounded height condition in this statement. Bounding the height of a poset is nothing else than excluding a long chain as a subposet. A natural line of research is concerned with $(k+k)$-free posets ($k \geqslant 2$), which are defined by excluding two totally incomparable chains of length k as a subposet. Another natural restriction is to exclude the standard example S_k as a subposet (again $k \geqslant 2$). In particular, $(2+2)$-free posets and S_2-free posets are exactly interval orders.

(F) *Do planar $(k+k)$-free or S_k-free posets have dimension bounded in terms of k?*

The last thread of research we propose within the project concerns queue and stack layouts. A *queue layout* of a graph is an ordering of its vertices (the spine) together with an edge coloring such that there are no two nested monochromatic edges, where two edges are *nested* if all four endpoints are distinct and the endpoints of one edge induce an interval on the spine containing the endpoints of the other edge. A *queue layout* of a poset is a queue layout of its cover graph with an extra restriction that the ordering of vertices on the spine must be a linear extension of the poset. Then the *queue number* of a graph (poset) is the minimum number of colors in its queue layout. A *stack layout* of a graph is an ordering of its vertices (the spine) together with an edge coloring such that there are no two crossing monochromatic edges, where two edges are *crossing* if all four endpoints are distinct and the endpoints of one edge altenate with the endpoints of the other on the spine. A *stack layout* of a poset is a stack layout of its cover graph with an extra restriction that the ordering of vertices on the spine must be a linear extension of the poset. Then the *stack number* of a graph (poset) is the minimum number of colors in its stack layout.

Graphs with small queue/stack layouts attract our attention as they form a class of graphs that allows any graph as a minor but has bounded expansion (in the sense of Nešetřil-Ossona de Mendez sparsity theory, see subsection 2.2). In particular, our results from [24] imply that posets with bounded height and bounded queue/stack number have bounded dimension. We believe that techniques developped in that work might be useful to attack some longstanding open problems concerning graph/poset layouts. Probably the most exciting challenge on the graph-theoretic side is to verify the following:

(G) *Is there a constant c such that* $\text{queue}(G) \leqslant c$ *for planar graphs G?*

This problem has a rich literature and a positive resolution would have important implications for graph drawing (see subsection 2.3). On the poset-theoretic side, the following questions are of our interest:

(H) *Is there a constant c such that* $\text{stack}(P) \leqslant c$ *for planar posets P?*
(I) *Improve the bounds on* $\text{queue}(P)$ *in terms of* $\text{width}(P)$*,* $\text{height}(P)$ *and* $|P|$ *for planar posets P.*

Within the last problem we want to focus on three conjectures of Heath and Pemmaraju [17]: (1) whether $\text{queue}(P) \leqslant \text{width}(P)$ for any planar poset P; (2) whether $\text{queue}(P) = O(\sqrt{n})$ for any planar poset P of size n; (3) whether $\text{queue}(P) \leqslant \text{height}(P)$ for any planar poset P.

2. Significance

Partially ordered sets, *posets* for short, are studied extensively in combinatorics and theoretical computer science. The *dimension* of a poset P, denoted by $\dim(P)$, is the minimum integer d such that the points of P can be embedded into $\mathbb{R}^d$ in such a way that $x < y$ in P if and only if $x_i < y_i$ for every coordinate $i = 1, \ldots, d$, that is, P is isomorphic to the product order on the set of points in $\mathbb{R}^d$ to which it is mapped. Equivalently, the dimension of P is the minimum d such that there are d linear extensions of P whose intersection gives rise to P.

The notion of dimension has been introduced in 1941 by Dushnik and Miller [8] and proved its importance over the years. It has a clear computer science flavor, namely, for a given poset P with n elements and dimension d, it is enough to store d linear extensions witnessing the dimension of P in order to answer queries about the order relation of any two given elements x and y of P in time $O(d)$. This requires $O(dn)$ space instead of $\Omega(n^2)$ that would be necessary to store the entire matrix of comparabilities for a general poset with n points. Classes of posets for which the dimension can be bounded by a constant are particularly interesting in this context.

Poset dimension is closely related to the reachability problem in directed graphs, which is to design an efficient data structure to answer queries of the form "is there a directed path from u to v in the graph?". This relation, after contracting each strongly connected component to a single vertex, defines a partial order, whose cover graph is a minor of the original graph in its undirected version. Representing this poset by d linear extensions as it is explained above yields a data structure of size $O(dn)$ with query time $O(d)$ for the reachability problem. The reachability problem is particularly well studied for planar graphs, for which an algorithm due to Thorup [42] achieves constant query time with a data structure of size $O(n \log n)$, and for a special case of which an algorithm due to Kameda [26] achieves constant query time with a data structure of size $O(n)$. Actually, Kameda's algorithm implicitly uses the fact that every planar poset with a minimum and a maximum element has dimension at most 2.

The dimension of posets exhibits similar combinatorial and algorithmic properties to the chromatic number of graphs. For instance, there is a very simple construction of posets with dimension d, so-called *standard example* S_d (see Figure 2), but also there are posets of arbitrarily large dimension with no subposets isomorphic to S_3. For another example, testing whether the dimension is at most a fixed constant d is polynomial for $d = 2$ and NP-complete for $d \geqslant 3$ (see the following subsection).

The dimension also has unexpected connections. Perhaps the most striking example is that a graph G is planar if and only if the incidence poset of G (the bipartite poset on $V(G) \cup E(G)$ such that $v < e$ whenever vertex v is an endpoint of edge e) has dimension at most 3. This non-trivial characterization of planar graphs was proved by Schnyder [39] via colorings and orientations of edges of planar graphs nowadays known as Schnyder woods, which became an independent object of interest.

For a thorough introduction to dimension theory we refer to Trotter's book [43] and his chapter in *Handbook of Combinatorics* [44].

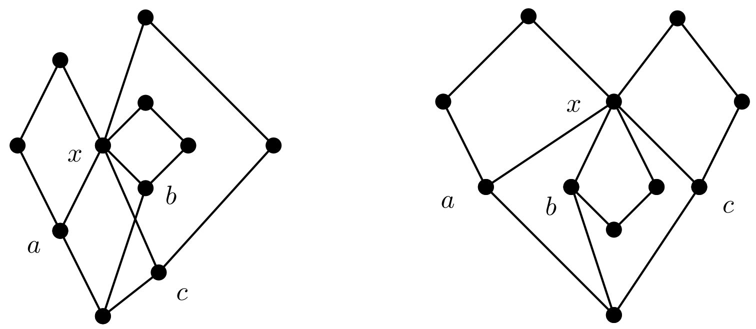

FIGURE 1. Poset with no planar diagram and a planar cover graph. Diagram on the left, cover graph drawn on the right.

2.1. **Computational complexity.** In spite of its central importance in the theory of posets, understanding of the computational aspects of dimension has lagged behind that of its combinatorial properties and of its relation to other parameters of posets. The computational complexity of determining the dimension of a poset was one of the twelve outstanding open problems in Garey and Johnson's treatise on NP-completeness [14]. It has been proved NP-complete independently by Lawler and Vornberger [30] and by Yannakakis [48]. Actually, Yannakakis proved the stronger statement that deciding whether the dimension is at most d is NP-complete for every fixed $d \geqslant 3$, using a reduction from graph 3-colorability. On the other hand, the results of Dushnik and Miller's pioneering work [8] easily imply that one can test in polynomial time whether the dimension is at most 2 (McConnell and Spinrad [31] gave a linear-time algorithm). Spinrad [40] gives an account of several other computational aspects of poset dimension theory.

Approximating the dimension is also hard. Chalermsook, Laekhanukit, and Nanongkai [3] showed that unless NP = ZPP no polynomial time algorithm exists that approximates the dimension of a poset within a factor of $O(n^{1-\varepsilon})$ for any $\varepsilon > 0$, which improves an earlier result of Hegde and Jain [21] on hardness of $O(n^{1/2-\varepsilon})$-approximation.

How about parameterized algorithms for dimension? The two natural choices for parameters are the height and the width of the poset. For the height, Yannakakis [48] showed that deciding whether the dimension of a poset of height 2 has dimension at most d is NP-complete for every fixed $d \geqslant 4$. The decision problem for posets of height 2 whether they have dimension at most 3 was open over 30 years and was recently proved to be NP-complete as well, by Felsner, Mustaţă and Pergel [10].

On the other hand, the question about complexity of computing dimension for posets of bounded width, that is, problem (**A**), seems to be a true mystery. Möhring [33] proposed this question already in the 1980s. Since $\dim(P) \leqslant \text{width}(P)$ and deciding whether $\dim(P) \leqslant 2$ is polynomial, the question starts to be interesting for posets P with $\text{width}(P) \geqslant 4$. The NP-hardness proof of by Yannakakis does not cover this case, as it uses partial orders whose width grows with the size of the instance.

We would like to approach problem (**A**) from the point of view of parameterized complexity theory. A problem with parameter k is called *fixed-parameter tractable* (*FPT* in short) if it can be decided in time bounded by $f(k)n^{\alpha}$ for some (computable) function f of the parameter and some absolute constant α. Problem (**B**) is nothing else but the question: "Is computing the dimension of a poset of width at most w FPT when parameterized by w?" Clearly, a positive answer to question (**A**) would imply a positive answer to question (**B**).

2.2. **Sparsity and dimension.** A standard way of visualizing posets is by their *diagrams*: the points of a poset are placed in the plane and whenever $a < b$ in the poset and there is no point c with $a < c < b$, a curve is drawn from a to b going upwards (y-monotone). The diagram thus represents *cover relations* of the poset, that is, those relations which are minimal in the sense that they are not implied by transitivity. An alternative way of understanding the diagram, which abstracts from geometry, is to draw it as a directed graph in which a cover relation $a < b$ is represented by the directed edge $a \to b$. The undirected graph defined by a diagram (in either sense) is the *cover graph* of the poset. For some concepts it may be important which of the two notions of a diagram is in use. For example, not every poset with planar cover graph (planar diagram in the second sense) has a planar y-monotone drawing (planar diagram in the first sense), see Figure 1. Posets that have a planar y-monotone drawing are called themselves *planar*. Thus, when considering planarity in this project, we usually mean planarity of the drawing, as it gives us additional geometric tools to study the problem. However, for more general study of posets with bounded 'topology', we are concerned with posets whose cover graphs belong to some specific sparse classes of graphs, thus implicitly assuming the second understanding of a diagram.

There is a common belief that a poset having a 'nice drawing' or 'not too dense' cover graph should have small dimension. In this vein, Trotter and Moore [45] showed that if the cover graph of a poset P is a forest,

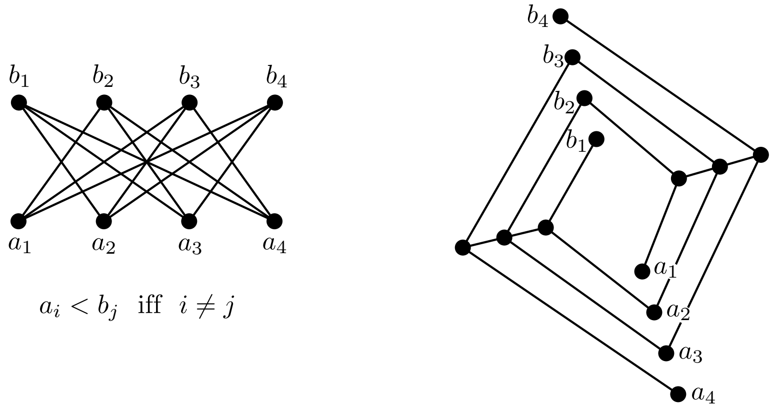

FIGURE 2. Standard example S_4 (left) and Kelly's construction of a planar poset containing S_4 (right). The *standard example* S_d is the poset on $2d$ points consisting of d minimal points $a_1, \dots, a_d$ and d maximal points $b_1, \dots, b_d$ such that $a_i < b_j$ in S_d if and only if $i \neq j$. It is well known and easy to verify that $\dim(S_d) = d$. For every $d \geqslant 1$, Kelly's construction provides a planar poset with $4d-2$ points containing S_d as a subposet and hence having dimension at least d; its general definition is implicit in the figure.

then $\dim(P) \leqslant 3$, and this bound is best possible. However, already for planar cover graphs one cannot hope for the dimension to be constant. In 1981, Kelly [27] presented a family of planar posets with arbitrarily large dimension, see Figure 2. In view of Kelly's construction, the study of connections between the dimension and the structure of the cover graph was withheld for almost 30 years.

Recently, this study has been resumed with the general purpose of determining which properties of Kelly's construction are essential for obtaining posets with sparse cover graphs and arbitrarily large dimension. Felsner, Trotter and Wiechert [11] proved that if the cover graph of a poset P is outerplanar, then $\dim(P) \leqslant 4$. Again, this bound is best possible. Biró, Keller and Young [2] showed that if the cover graph of a poset P has pathwidth at most 2, then $\dim(P) \leqslant 17$. Finally, Joret, Micek, Trotter, Wang and Wiechert [23] proved the most general statement of this kind so far—that if the cover graph of a poset P has treewidth at most 2 (which is equivalent to excluding K_4 as a minor), then $\dim(P) \leqslant 1276$. This constitutes a background for objective (**C**), where we would like to characterize the minor-closed classes of graphs that guarantee constant dimension. For instance, Kelly's construction implies that it is not enough to exclude K_5, but our latter result asserts that excluding K_4 suffices. It is worth noting that objective (**C**) harmonizes very well with the application of poset dimension to reachability described earlier in this section, as the cover graph of the order obtained by contracting strongly connected components considered there is a minor of the original graph. Therefore, for any minor-closed class of graphs $\mathscr{C}$, a positive answer to the question in (**C**) yields a data structure with size $O(n)$ and query time $O(1)$ for the reachability problem on directed graphs whose undirected versions belong to $\mathscr{C}$. We hope to obtain some interesting new results on the reachability problem as corollaries to the results of this project.

The next key observation about Kelly's constructions is that the dimension of these posets grow together with their height. This leads us to a stream of research and results over last three years which takes its roots in the following breakthrough result of Streib and Trotter [41]: posets with planar cover graphs have dimension bounded by a function of height. Due to extensive use of Ramsey theory, the bounding function in that result is enormous. Within a team (Joret, Micek, Milans, Trotter, Walczak, and Wang [22]), we proved that posets with cover graphs of bounded treewidth have dimension bounded by a function of height. This combined with a simple reduction from [41] yields an analogous statement for cover graphs excluding a fixed apex graph as a minor, which generalizes the result on planar graphs. Furthermore, the proof is much simpler (in particular, it avoids the use of Ramsey theory) and yields a double exponential bound on the dimension in terms of the height for planar cover graphs. We are very positive about the possibility of further improvement of this function, which is our objective (**D**).

In another direction, Füredi and Kahn [13] showed that posets with cover graphs of bounded maximum degree have dimension bounded in terms of their height.[1] All these results for planar, bounded treewidth and bounded degree cover graphs were very recently generalized by Walczak [47, SODA 2015]: posets whose cover graphs exclude a fixed graph as a topological minor have dimension bounded by a function of their height. Walczak's proof used the ideas from [22] together with Robertson-Seymour and Grohe-Marx graph

[1] We note that the original statement of Füredi and Kahn's theorem is that posets with *comparability* graphs of bounded maximum degree have bounded dimension; in fact, they show a $O(\Delta \log^2 \Delta)$ upper bound on the dimension, where Δ denotes the maximum degree. Observe however that the comparability graph of a poset has bounded maximum degree if and only its cover graph does and its height is bounded.

structure theorems for classes of graphs with an excluded topological minor. Together with Wiechert [32], we gave an alternative elementary proof, again improving significantly the bound on the dimension in terms of the height.

A natural continuation of this line of research is to investigate the sparse classes of graphs along the hierarchy introduced by Nešetřil and Ossona de Mendez [35]. In particular, they introduced classes of graphs with *bounded expansion* and *nowhere dense* classes of graphs. The key idea in both cases is to look at minors of *bounded depth*, that is, minors that can be obtained by first contracting disjoint connected subgraphs of bounded radius, and then possibly removing some vertices or edges. In a nowhere dense class it is required that bounded-depth minors exclude at least one graph (which can depend on the depth). In a class with bounded expansion the requirement is stronger: for every $r \geqslant 1$, the minors of depth r should be sparse, that is, their average degrees should be bounded by some function $f(r)$. It is relatively easy to verify that every class of graphs excluding some graph as a topological minor has bounded expansion.

Together with Joret and Wiechert [24, submitted to SODA 2016], we proved that for any class of graphs $\mathscr{C}$ with bounded expansion (see further on for the definition of bounded expansion), posets with cover graphs in $\mathscr{C}$ have dimension bounded by a function of height. This generalizes all previous results asserting such a bound. We also showed that bounded expansion cannot be replaced by bounded degeneracy or by the property of being nowhere dense. Therefore, in a way, we have characterized the types of graph sparsity which ensure that the dimension is bounded by a function of height. Objective (**E**) attempts to save some meaningful bound on dimension for these very broad but still sparse classes of graphs. If the answer to (**E**) is positive (which we strongly believe in), this would be one more evidence for the meta-statement that *the dimension for posets behaves like the chromatic number for graphs*, as the chromatic number of graphs from a nowhere dense class is indeed $O(n^{\varepsilon})$ [34].

The result of [24] on classes of bounded expansion has some nice consequences in the context of graph drawing, where natural classes with bounded expansion appear that do not fit in the setting of excluding a topological minor. They include k-planar graphs (graphs that can be drawn in the plane with at most k crossings per edge) and graphs with bounded queue or stack layouts, which we discuss later on [36]. This has brought our attention to the problems listed as objectives (**G**)–(**I**), which are discussed in more detail in the next subsection.

In Figure 3, we give a summary of all the known results about posets with sparse cover graphs having their dimension bounded by a constant or by a function of their height.

Bounding the height of a poset is the same as excluding a chain of a fixed length as a subposet. So far, for classes of graphs $\mathscr{C}$ such that posets with cover graphs in $\mathscr{C}$ can have arbitrarily large dimension, only bounding the height has been considered as an additional restriction to get a constant bound. But it is not necessarily the most natural restriction—one can investigate how the dimension behaves if some other structure contained in Kelly's construction becomes excluded. Objective (**F**) is the first natural step in this direction, where we propose to exclude $k+k$ (two totally incomparable chains of length k) as a subposet instead of bounding the height. Exclusion of $k+k$ has led to meaningful results in the context of on-line dimension [9]. Another natural structure that one could exclude instead of bounding the height is the standard example S_k—studying the dimension of S_k-free posets is like studying the chromatic number of K_k-free graphs. It is conceivable that all the above-mentioned bounds on the dimension in terms of the height can be generalized to bounds in terms of k for $(k+k)$-free or S_k-free posets. Note that both $(k+k)$-free and S_k-free posets provide a natural generalization of interval orders, which are the same as $(2+2)$-free posets and the same as S_2-free posets.

2.3. **Queue and stack layouts.** Queue layouts have been introduced by Heath, Leighton and Rosenberg [16, 20] in 1992, and have been extensively studied since. They have applications to VLSI design, parallel process scheduling, fault-tolerant processing, matrix computations on data-driven networks, and sorting networks (see [38] for an overview). There is also a remarkable application in graph drawing: a *3D grid drawing* of a graph is a placement of the vertices at distinct points in $\mathbb{Z}^3$, such that the line segments representing the edges are pairwise non-crossing; now, an n-vertex graph G has an $O(1) \times O(1) \times O(n)$ 3D grid drawing if and only if G has queue number $O(1)$ [7].

It is a quite celebrated problem in graph drawing whether planar graphs have linear-volume 3D grid drawings. As we discussed, this would follow from a positive resolution of problem (**G**). Question (**G**) was raised already in the first papers on queue layouts [16, 20], so it stands open over 20 years. The first non-trivial upper bound $O(\log^2 n)$ on queue number of planar graphs was given by Di Battista, Frati and Pach [4] and that was improved to $O(\log n)$ by Dujmović [6] with a nice and way shorter argument.

The concept somehow dual to a queue layout is a stack layout also known as a *book embedding*. Stack layouts, introduced by Kainen [25] and Ollmann [37], find similar applications as queue layouts. The literature is rich of combinatorial and algorithmic contributions on the stack number of various classes of graphs. Here in contrast, already in 1989, Yannakakis [49] proved that every planar graph has stack number at most 4. Although the cover graph of a planar poset is planar so it has stack number at most 4, not all stack layouts of the cover

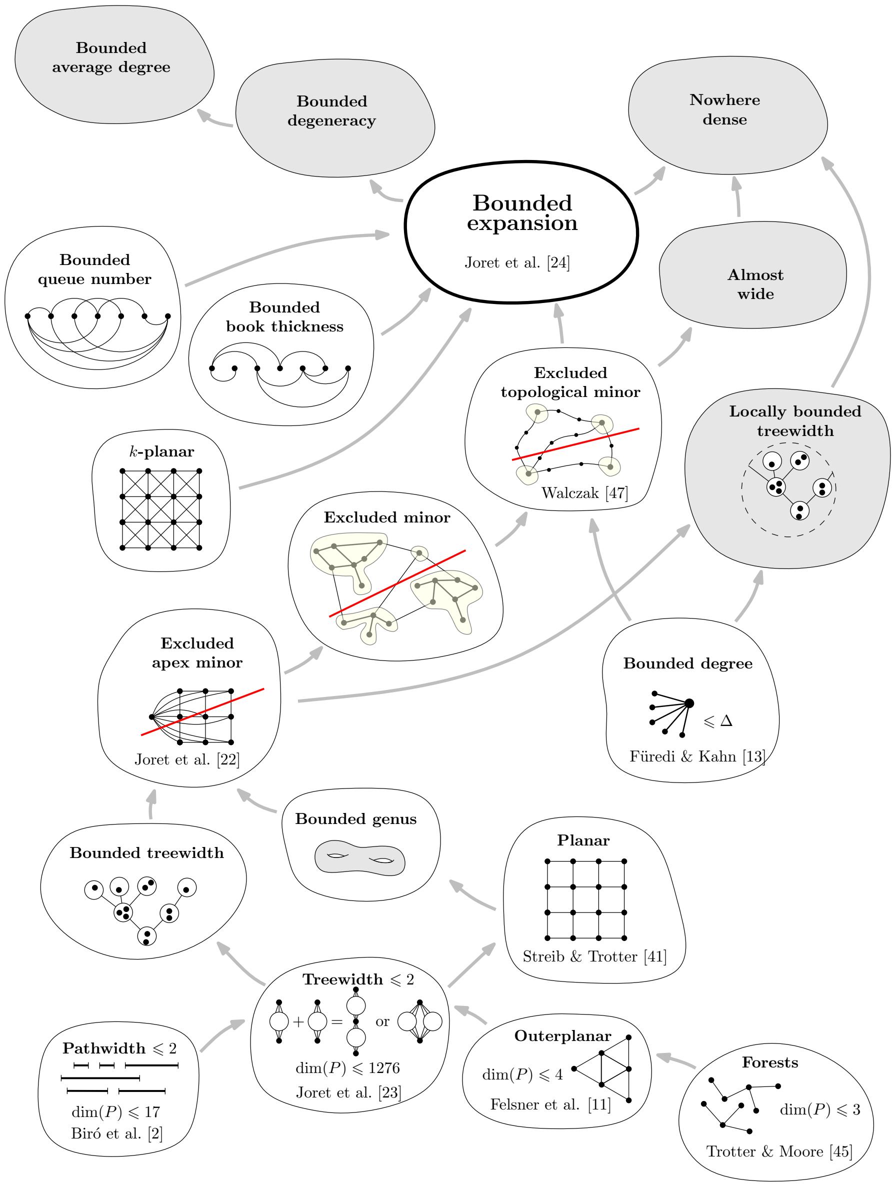

FIGURE 3. A hierarchy of sparse classes of graphs. A type of class is drawn in grey whenever it includes a class $\mathscr{C}$ such that there are posets of bounded height and unbounded dimension with cover graphs in $\mathscr{C}$.

graph are legal as stack layouts of the underlying poset. The question whether planar posets have bounded stack number remains open and constitutes objective (**H**). This problem was raised by Heath et al. and studied in [1, 5, 17, 18, 19]. It is known that when the poset diagram is a directed tree, then the stack number is 1 [19], and when the diagram is a planar 3-tree, then the stack number is bounded by a constant [12].

Objective (**I**) is inspired by Heath and Pemmaraju [17] who initiated the studies on stack and queue layouts for posets. They have shown that $\text{queue}(P) \leqslant \text{width}^2(P)$ and conjectured that $\text{queue}(P) \leqslant \text{width}(P)$ for all posets P. They constructed, for all n, planar posets P_n of size n with $\text{queue}(P_n) = \Omega(\sqrt{n})$ and conjectured that this is best possible. They also proved that for every planar poset P we have $\text{queue}(P) \leqslant 4\,\text{width}(P)$. Finally, they conjectured that $\text{queue}(P) \leqslant \text{height}(P)$ for all planar posets P.

3. WORK PLAN

Among the 9 objectives of the proposed project, there are problems that over the years proved themselves to be challenging and that the intended resolution would be recognized by the community. Below we provide our insights and source of hopes that we can make a progress in each task.

Concerning problem (**A**), first of all, we should note that there is no consensus in the community whether the answer should be ‘yes’ or ‘no’. Our idea to attack this problem is to consider it in the context of the fixed-parameter tractability when parameterized by the width, see problem (**B**). We would like to approach problem (**B**) restricting the input to specific but important classes of posets: interval orders, $(k+k)$-free posets, and posets with bounded interval dimension (i.e. multidimensional analogues of interval orders). The incomparability graphs of interval orders of bounded width have bounded pathwidth. Therefore, an approach to the problem for interval orders, which we would like to try first, is to use dynamic programming on the incomparability graph, which if successful would result in an FPT algorithm running in time $f(w)n$. Another possible relaxation of problems (**A**) and (**B**) is to give up exact computation of dimension and attempt for good approximation algorithms. Since the general problem of computing dimension is very hard to approximate (see section 2.1), this could yield interesting results in the context of posets of bounded width.

Concerning the family of problems (**C**)–(**F**), we are in a privileged position to attack them. Over last years and months, we produced a number of results in this area that generalize previous results but also very often simplify the arguments, so the key ideas are visible and ready to push forward. In this context, we would like to emphasise two results and papers finished this year (2015): [32, submitted to *Journal of Graph Theory*] and [24, submitted to SODA 2016], which have been already described and cited in section 2.2. They both use a very promising new technique that we call *unrolling of a poset*. On the level of intuitions it works as follows: if a poset has large dimension, then it has a 'local' subposet which still has large dimension. This type of statement has a very simple and descriptive analogue in the world of graphs: if a graph G is connected and $\chi(G) > 2k$, then at least one distance level L (considered from any fixed vertex v) satisfies $\chi(G[L]) > k$. The 'locality' stems from the fact that in many cases (e.g. in minor-closed classes of graphs) we can handle all the previous distance levels as if they were a single vertex. Extracting such levels in an iterative manner is a powerful tool when one tries to bound the chromatic number of graphs with geometric representations, see e.g. [15, 29]. We believe that we have just developed a poset counterpart of this method.

For problem (**C**), we do not have yet a guess what the exact characterization could look like, but our paper [23] with a proof that excluding K_4 as a minor is enough to get constant dimension contains a long argument full of independent ideas which hopefully can work with some other excluded minors. Our first candidate to investigate, for which we do not know yet if it is on the positive or the negative side, is the graph formed by a path (of fixed length) with one extra vertex adjacent to all vertices of the path. Large Kelly examples contain this graph as a minor so they do not witness a negative resolution like they do for the planar case.

Concerning problem (**D**), we already have an on-going research about it and we are very positive about the future progress. As we discussed, a possible positive resolution opens the question for even more exact bounds for planar case; these questions were very rare approachable in the past in dimension theory.

Our starting point to attack problem (**E**) on posets with cover graphs from a nowhere dense class is the argument from [24] dealing with posets with cover graphs from a class with bounded expansion. The very same argument applied directly to nowhere dense class of graphs is too weak and does not yield any interesting bound but at least we have a very suggestive picture which part of the proof we need to improve.

Concerning problem (**F**), we already have two partial results. The first one is that interval orders (that is, $(2+2)$-free or S_2-free posets) with cover graphs excluding any fixed graph as a minor (in particular, planar interval orders) have bounded dimension. This is a relatively easy consequence of a result due to Kierstead and Trotter [28] that for every poset Q, interval orders with dimension large enough contain subposets isomorphic to Q. Our second partial result is that for every p, $(k+k)$-free posets with cover graphs of pathwidth at most p have dimension bounded in terms of k and p. We would like to push forward this discovery and replace bounded pathwidth condition by bounded treewidth. This would be a major step towards the resolution of problem (**F**) for $(k+k)$-free posets.

Graphs with bounded queue number or bounded stack number form a class with bounded expansion (see [36]). Our initial interest in queue and stack layouts came from a suspicion that the techniques we developed for general classes of graphs with bounded expansion could be fine-tuned to give interesting results for posets with small queue or stack layouts. Since a number of open questions concerning these layouts address planar posets, we also hope to apply our machinery designed for problem (**D**). In particular, the question whether $\text{queue}(P) \leqslant \text{height}(P)$, which is part of problem (**I**), looks like a good start for us.

4. METHODOLOGY

We will use and further develop methods that have proven useful in previous research on poset dimension, algorithmic and structural graph theory (e.g. structure theorems for graphs with excluded minors) and queue/stack layouts, and in construction of FPT algorithms for restricted classes of graphs (e.g. dynamic programming), as well as search for new tools. Wherever we find it useful, we will run computer experiments to verify the validity of our hypotheses.

5. Appointing a new scientific team

The project's team consists of Principal Investigator Piotr Micek, postdoc Bartosz Walczak and two students to be selected in accordance with the Resolution of National Science Centre No. 50/2013 of 3 June 2013 on the granting of scholarships in NSC research projects.

Bartosz Walczak obtained his Ph.D. degree in 2012 has already significant experience in research on poset dimension. He has three papers on the topic [22, 46, 47], the first one joint with the P.I. He spent last academic year (2014–15) at Georgia Institute of Technology in Atlanta with William T. Trotter, who is one of the pioneers and remains a true leader of research in combinatorics of posets and in dimension theory specifically (see his book [43] and his chapter in *Handbook of Combinatorics* [44]). Before, Bartosz spent a year at EPFL in Lausanne with János Pach and then half a year at Charles University in Prague with Jan Kratochvíl.

The P.I. spent last two years at TU Berlin in Stefan Felsner's group. His very recent collaboration with Felsner's student Veit Wiechert resulted in three papers on poset dimension [23, 24, 32].

For three of the four years of the project's duration, we plan to complement the team with one Ph.D. student and one Master's student at each time. It is intended that each our student will focus primarily on a single research thread. We expect that the results of the project will give grounds for at least one Ph.D. thesis. Involving students in the project will also lead to efficient transfer of our knowledge and science practices.

We have no doubt that independent research experience and insights into the topic that the two of us (Piotr and Bartosz) have gained over the recent years complemented by fresh minds of our students will foster successful collaboration within the proposed project. Hopefully, it will also give rise to a strong research group on poset dimension theory at Jagiellonian University.

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